

The ring of modular forms for the even unimodular lattice of signature (2,10)

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Abstract

We show that the ring of modular forms with characters for the even unimodular lattice of signature (2,10) is generated by forms of weights 4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42, and 252 with one relation of weight 504. The main ingredient of the proof is the work of Shiga [Shi] on the moduli space of K3 surfaces polarized by the even unimodular lattice of signature (1,9).

1 Introduction

Let P be an even non-degenerate lattice of signature $(1, t)$ for some $0 \leq t \leq 19$. A P -polarized K3 surface is a pair (Y, j) of a K3 surface Y and a primitive lattice embedding $j : P \hookrightarrow \text{Pic } Y$. Lattice polarized K3 surfaces are introduced by Dolgachev [Dol96] to study mirror symmetry for K3 surfaces. The *mirror moduli space* of P -polarized K3 surfaces is the moduli space of \tilde{P} -polarized K3 surfaces, where $\tilde{P} = (P \perp U)^\perp$ is the orthogonal complement of the orthogonal sum of P and the even unimodular hyperbolic lattice U of rank 2 inside the K3 lattice $L = E_8 \perp E_8 \perp U \perp U \perp U$.

Let $T_{2,3,7}$ be the lattice determined by the Coxeter-Dynkin diagram shown in Figure 1.1. This lattice is isomorphic to $E_8 \perp U$ as an abstract lattice. This is the most symmetric lattice from the point of view of mirror symmetry, in the sense that the mirror dual lattice $\tilde{T}_{2,3,7}$ is isomorphic to the original one; $L \cong T_{2,3,7} \perp T_{2,3,7} \perp U$. Let M be the moduli space of $T_{2,3,7}$ -polarized K3 surfaces, which is isomorphic to the quotient \mathcal{D}/Γ of a symmetric domain \mathcal{D} of type IV by a discrete group Γ . Let further $T^* = \mathbf{P}(\mathbf{w})$ be the weighted projective space with weight

$$\mathbf{w} = (4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42). \quad (1.1)$$

The Satake-Baily-Borel compactification of M will be denoted by M^* . The following theorem is due to Shiga:

Theorem 1.1 (Shiga [Shi]). *There is an isomorphism $T^* \xrightarrow{\sim} M^*$ of algebraic varieties.*

Let $\mathbb{M} = [\mathcal{D}/\Gamma]$ and $\mathbb{P}(\mathbf{w}) = [(\mathbb{C}^{11} \setminus \mathbf{0})/\mathbb{C}^\times]$ be the orbifold quotients, whose coarse moduli spaces are $M = \mathcal{D}/\Gamma$ and $T^* = \mathbf{P}(\mathbf{w})$ respectively. For a pair of an orbifold and a divisor on it, one can perform the *root construction* [AGV08, Cad07] to introduce a generic stabilizer along the divisor. Let \mathbb{T}^* be the orbifold obtained from $\mathbb{P}(\mathbf{w})$ by the root construction of order two along a divisor \mathbb{H}^* of order 504. The main result of this paper is the following:

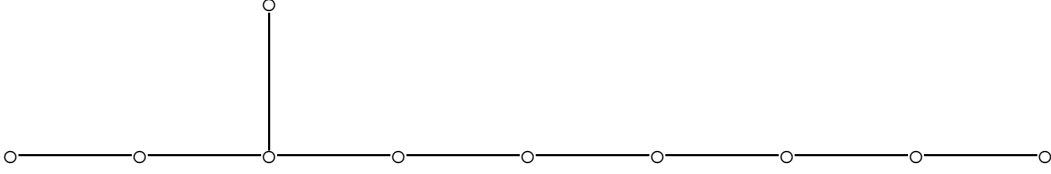


Figure 1.1: The Coxeter-Dynkin diagram of the lattice $T_{2,3,7} = E_8 \perp U$

Theorem 1.2. *There is a bimeromorphic map $\mathbb{T}^* \dashrightarrow \mathbb{M}$ of orbifolds, which is an isomorphism in codimension one.*

This is a refinement of Theorem 1.1, whose proof follows that of Shiga [Shi] closely. The orbifold \mathbb{T}^* has singularities coming from the singularities of the divisor \mathbb{H}^* , whereas \mathbb{M} is smooth.

A modular form of weight k and character $\chi \in \text{Char}(\Gamma) = \text{Hom}(\Gamma, \mathbb{C}^\times)$ is a holomorphic function $f : \mathcal{D} \rightarrow \mathbb{C}$ satisfying

- (i) $f(\lambda z) = \lambda^{-k} f(z)$ for any $\lambda \in \mathbb{C}^\times$, and
- (ii) $f(\gamma z) = \chi(\gamma) f(z)$ for any $\gamma \in \Gamma$.

The vector spaces $A_k(\Gamma, \chi)$ of modular forms constitute the ring

$$A(\Gamma) = \bigoplus_{k=0}^{\infty} \bigoplus_{\chi \in \text{Char}(\Gamma)} A_k(\Gamma, \chi)$$

of modular forms. As a corollary to Theorem 1.2, one obtains the following structure theorem of the ring of modular forms:

Corollary 1.3. *The ring $A(\Gamma)$ of modular forms is generated by forms of weights 4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42, and 252 with one relation of weight 504.*

The relation in Corollary 1.3 is the defining equation of the divisor $\mathbb{H}^* \subset \mathbb{P}(\mathbf{w})$. Its description as the ratio of a discriminant and a resultant is given in Section 4.

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2 Lattice polarized K3 surfaces

Let $L = E_8 \perp E_8 \perp U \perp U \perp U$ be the K3 lattice, and $P = T_{2,3,7}$ be the even unimodular lattice of signature $(1, 9)$ appearing in Introduction. Fix a connected component $V(P)^+$ of $V(P) = \{x \in P_{\mathbb{R}} \mid (x, x) > 0\}$ and choose a subset $\Delta(P)^+$ of $\Delta(P) = \{\delta \in P \mid (\delta, \delta) = -2\}$ satisfying

1. $\Delta(P) = \Delta(P)^+ \amalg (-\Delta(P)^+)$ and
2. $\Delta(P)^+$ is closed under addition (but not subtraction).

Define $C(P)^+ = \{h \in V(P)^+ \cap P \mid (h, \delta) > 0 \text{ for any } \delta \in \Delta(P)^+\}$ and set

$$\begin{aligned}\text{Pic}(Y)^+ &= C(Y) \cap H^2(Y; \mathbb{Z}), \\ \text{Pic}(Y)^{++} &= C(Y)^+ \cap H^2(Y; \mathbb{Z}),\end{aligned}$$

where $C(Y)^+ \subset H^{1,1}(Y) \cap H^2(Y; \mathbb{R})$ is the Kähler cone of Y and $C(Y)$ is its closure.

Definition 2.1 (Dolgachev [Dol96]). A P -polarized K3 surface is a pair (Y, j) where Y is a K3 surface and $j : P \hookrightarrow \text{Pic}(Y)$ is a primitive lattice embedding. An *isomorphism* of P -polarized K3 surfaces (Y, j) and (Y', j') is an isomorphism $f : Y \rightarrow Y'$ of K3 surfaces such that $j = f^* \circ j'$. An P -polarized K3 surface is *pseudo-ample* if $j(C(P)^+) \cap \text{Pic}(Y)^+ \neq \emptyset$, and *ample* if $j(C(P)^+) \cap \text{Pic}(Y)^{++} \neq \emptyset$.

Fix a primitive lattice embedding $i_P : P \hookrightarrow L$ and let $Q = P^\perp$ be the orthogonal lattice, which is isomorphic to $P \perp U$ as an abstract lattice. The period domain \mathcal{D} is a connected component of $\{[\Omega] \in \mathbf{P}(Q \otimes \mathbb{C}) \mid (\Omega, \Omega) = 0, (\Omega, \overline{\Omega}) > 0\}$, which is a bounded Hermitian domain of type IV. The global Torelli theorem and the surjectivity of the period map shows that the coarse moduli space of pseudo-ample P -polarized K3 surfaces is given by $M = \mathcal{D}/\Gamma$, where $\Gamma = O(Q)^+$ is the index two subgroup of the orthogonal group of the lattice Q preserving the connected component \mathcal{D} . The moduli space of ample P -polarized K3 surfaces is the subspace \mathcal{D}°/Γ of M , where $\mathcal{D}^\circ = \mathcal{D} \setminus H_{\mathcal{D}}$ is the complement of the union $H_{\mathcal{D}} = \bigcup_{\delta \in \Delta(Q)} \delta^\perp$ of reflection hyperplanes $\delta^\perp = \{[z] \in \mathbf{P}(Q \otimes \mathbb{C}) \mid (z, \delta) = 0\}$.

3 Moduli space of $T_{2,3,7}$ -polarized K3 surfaces

Let $\mathbf{P} = \mathbf{P}(6, 14, 21, 1)$ be the weighted projective space of weight $(6, 14, 21, 1)$ and consider the family

$$\varphi_{\tilde{T}^*} : \tilde{\mathfrak{Y}}^* = \left\{ ([x : y : z : w], t) \in \mathbf{P} \times \tilde{T}^* \mid f(x, y, z, w; t) = 0 \right\} \rightarrow \tilde{T}^* \quad (3.1)$$

of hypersurfaces of \mathbf{P} , where

$$f(x, y, z, w; t) = z^2 + y^3 + g_2(x, w; t)y + g_3(x, w; t), \quad (3.2)$$

$$\begin{aligned}g_2(x, w; t) &= t_4 x^4 w^4 + t_{10} x^3 w^{10} + t_{16} x^2 w^{16} + t_{22} x w^{22} + t_{28} w^{28}, \\ g_3(x, w; t) &= x^7 + t_{12} x^5 w^{12} + t_{18} x^4 w^{18} + t_{24} x^3 w^{24} + t_{30} x^2 w^{30} + t_{36} x w^{36} + t_{42} w^{42},\end{aligned} \quad (3.3)$$

$$t = (t_4, t_{10}, t_{12}, t_{16}, t_{18}, t_{22}, t_{24}, t_{28}, t_{30}, t_{36}, t_{42}) \in \tilde{T}^* = \mathbb{A}^{11} \setminus \mathbf{0}. \quad (3.4)$$

The group \mathbb{C}^\times acts on \mathbf{P} and \tilde{T}^* in such a way that $\alpha \in \mathbb{C}^\times$ sends $[x : y : z : w] \in \mathbf{P}$ to $[x : y : z : \alpha^{-1}w]$ and $t = (t_i)_{i=4}^{42}$ to $\alpha \cdot t = (\alpha^i t_i)_{i=4}^{42}$. Since f is invariant under the \mathbb{C}^\times -action, the family $\varphi_{\tilde{T}^*} : \tilde{\mathfrak{Y}} \rightarrow \tilde{T}^*$ descends to a family $\varphi_{T^*} : \mathfrak{Y} \rightarrow T^*$ over the weighted projective space $T^* = \mathbf{P}(\mathbf{w})$ with weight

$$\mathbf{w} = (4, 10, 12, 16, 18, 22, 24, 28, 30, 36, 42). \quad (3.5)$$

The fiber of $\varphi_{T^*} : \mathfrak{Y} \rightarrow T^*$ over $[t] \in T^*$ will be denoted by \overline{Y}_t . Let T be the set of $[t] \in T^*$ such that \overline{Y}_t has at worst rational double points. The following fact is well-known:

Proposition 3.1 (cf. e.g. [Mir89, Proposition III.3.2]). *An elliptic surface of the form*

$$z^2 + y^3 + g_2(x)y + g_3(x) = 0 \quad (3.6)$$

has a singularity worse than rational double points in the fiber above $x = a$ if and only if $\text{ord}_a(g_2) \geq 4$ and $\text{ord}_a(g_3) \geq 6$.

It follows that \overline{Y}_t has a singularity worse than rational double points if and only if one can set

$$\begin{aligned} g_2(x, w) &= ax^4w^4, \\ g_3(x, w) &= bx^6w^6 + w^7 \end{aligned} \quad (3.7)$$

by a change of coordinates. By a simple change of coordinate from (3.7) to (3.3), one obtains the following:

Corollary 3.2. *The complement $T^* \setminus T$ consists of points parametrized as*

$$t = [a : -4ab : -21b^2 : 6ab : 70b^3 : -4ab^3 : -105b^4 : ab^4 : 84b^5 : -35b^6 : 6b^7] \quad (3.8)$$

for $[a : b] \in \mathbf{P}(4, 6)$.

The adjunction formula shows that \overline{Y}_t has the trivial canonical sheaf. Since the minimal resolution of a surface is crepant if and only if it has at worst rational double points, one obtains the following:

Corollary 3.3. *The minimal model Y_t of \overline{Y}_t is a K3 surface if and only if $t \in T$.*

Let $P = T_{2,3,7} = E_8 \perp U$ be the lattice appearing in Introduction.

Proposition 3.4. *The minimal model Y_t for $[t] \in T$ has a natural structure of a pseudo-ample P -polarized K3 surface.*

Proof. The divisor $Y_\infty = Y \cap \{w = 0\}$ at infinity is given by

$$Y_\infty = \{[x : y : z] \in \mathbb{P}(6, 14, 21) \mid x^7 + y^3 + z^2 = 0\}, \quad (3.9)$$

which is a rational curve. The hypersurface \overline{Y}_t has

- an A_6 -singularity at $[0 : -1 : 1 : 0]$,
- an A_2 -singularity at $[-1 : 0 : 1 : 0]$, and
- an A_1 -singularity at $[1 : -1 : 0 : 0]$,

all coming from the singularity of the ambient space $\mathbb{P}(1, 6, 14, 21)$. The minimal resolution $Y_t \rightarrow \overline{Y}_t$ is a K3 surface with a configuration of (-2) -curves whose dual intersection graph is given by $T_{2,3,7}$. \square

We call the singularities appearing in the proof of Proposition 3.4 as *generic singularities of the family*.

Proposition 3.5. *The P -polarized K3 surface Y_t for $[t] \in T$ is ample if and only if \overline{Y}_t has only generic singularities of the family.*

Proof. A P -polarized K3 surface Y_t is strictly pseudo-ample if and only if its period Ω is on the reflection hyperplane H_δ of a root $\delta \in \Delta(Q)$. This happens if and only if the element δ considered as a cohomology class by the embedding $Q \subset H^2(Y_t; \mathbb{Z})$ induced by the P -polarization is Poincaré dual to a (-2) -curve, since the orthogonal lattice of Ω inside $H^2(Y_t; \mathbb{Z})$ is the Néron-Severi lattice. This (-2) -curve is contracted in \overline{Y}_t since δ is orthogonal to P , so that it appears as a singularity of \overline{Y}_t , which must be distinct from generic singularities in the family since the class δ is not contained in P . \square

Proposition 3.6. *For any pseudo-ample P -polarized K3 surface Y , there exist $[t] \in T$ and an isomorphism $Y \xrightarrow{\sim} Y_t$ of pseudo-ample P -polarized K3 surfaces.*

Proof. We identify P with its image by $j: P \hookrightarrow \text{Pic}(Y)$. Choose a basis $\{e, f\}$ of the orthogonal summand U of $P = U \perp E_8$ in such a way that $(e, e) = (f, f) = (e, f) - 1 = 0$ and $f \in C(P)$. Pseudo-ameness of Y implies that f is nef. Then one can show (cf. [PŠŠ71, §3, Theorem 1]) that Y admits a unique structure of an elliptic K3 surface with a section such that f is the class of a fiber and $e - f$ is the class of a section.

An elliptic K3 surface with a section admits a Weierstrass model of the form

$$z^2 = y^3 + g_2(x, w)y + g_3(x, w) \quad (3.10)$$

in $\mathbb{P}(1, 4, 6, 1)$ (cf. e.g. [SS10, Section 4]). Since the sublattice $E_8 \subset P$ is orthogonal to $f \in U$, it is generated by irreducible components of a fiber of Kodaira type II^* . One can choose a coordinate in such a way that this fiber lies over the point $x = \infty$ (or $w = 0$) in \mathbb{P}^1 . In order for the elliptic surface (3.10) to have a singular fiber of type II^* at ∞ , one needs

$$\text{ord}_\infty g_2(x, w) \geq 4, \quad \text{ord}_\infty g_3(x, w) = 5, \quad \text{and} \quad \text{ord}_\infty \Delta(x, w) = 10, \quad (3.11)$$

where $\Delta = 4g_2^2 - 27g_3^3$ (cf. e.g. [Mir89, Table IV.3.1]). This requires

$$g_2(x, w) = \sum_{i=0}^4 s_i x^i w^{8-i}, \quad (3.12)$$

$$g_3(x, w) = \sum_{i=0}^7 t_i x^i w^{12-i}. \quad (3.13)$$

If $t_7 = 0$, then Y has a singularity worse than rational double point. Hence one can set $t_7 = 1$ and $t_6 = 0$ by a change of coordinates of (x, w) . A birational change

$$\left(\frac{x}{w}, \frac{y}{w^4}, \frac{z}{w^6} \right) \mapsto \left(\frac{x}{w^6}, \frac{y}{w^{14}}, \frac{z}{w^{21}} \right)$$

of the ambient space from $\mathbf{P}(1, 4, 6, 1)$ to $\mathbf{P}(6, 14, 21, 1)$ sends (3.10) to (3.1), and Proposition 3.6 is proved. \square

The proof of Proposition 3.6 also shows the following:

Proposition 3.7. *For an isomorphism $\phi: Y_t \rightarrow Y_{t'}$ of pseudo-ample P -polarized K3 surfaces, there exists an element $\alpha \in \mathbb{C}^\times$ such that the following diagram commutes;*

$$\begin{array}{ccccc} Y_t & \xrightarrow{\varphi} & \overline{Y}_t & \xrightarrow{\iota} & \mathbf{P}(6, 14, 21, 1) \\ \phi \downarrow & & & & \downarrow \phi_\alpha \\ Y_{t'} & \xrightarrow{\varphi'} & \overline{Y}_{t'} & \xrightarrow{\iota'} & \mathbf{P}(6, 14, 21, 1). \end{array} \quad (3.14)$$

Here φ and φ' are minimal resolutions, ι and ι' are inclusions, and ϕ_α is the automorphism of $\mathbf{P}(6, 14, 21, 1)$ sending $[x : y : z : w]$ to $[x : y : z : \alpha w]$.

Proof. The proof of Proposition 3.6 shows that the embeddings ι and ι' given by the Weierstrass models (3.1) are determined by the pseudo-ample P -polarization up to an automorphism of $\mathbf{P}(1, 6, 14, 21)$. The automorphism group of $\mathbf{P}(1, 6, 14, 21)$ consists of transformations of the form

$$x \mapsto \beta_1 x + \beta_2^6 w, \quad (3.15)$$

$$y \mapsto \gamma_1 y + \gamma_2 x w^8 + \gamma_3 x^2 w^2, \quad (3.16)$$

$$z \mapsto \delta_1 z + \delta_2 x y w + \delta_3 x w^{15} + \delta_4 x^2 w^9 + \delta_5 x^3 w^3, \quad (3.17)$$

$$w \mapsto \alpha w. \quad (3.18)$$

The only automorphism which preserves the Weierstrass model (3.1) is $w \mapsto \alpha w$. \square

The discussions so far is summarized as follows:

Theorem 3.8 ([Shi]). *The period map gives an isomorphism $\Pi: T \rightarrow M$.*

4 Orbifold structure in codimension one

Let \mathcal{Q} be the connected component of $\{\Omega \in Q \otimes \mathbb{C} \mid (\Omega, \Omega) = 0, (\Omega, \overline{\Omega}) > 0\}$ which projects to \mathcal{D} . The natural projection $\pi: \mathcal{Q} \rightarrow \mathcal{D}$ is a principal \mathbb{C}^\times -bundle over \mathcal{D} , which is trivial since it admits a section

$$\{e - (v, v)/2f + v \in \mathcal{Q} \mid v = v_1 + \sqrt{-1}v_2 \in P \otimes \mathbb{C}, (v_2, v_2) > 0\}. \quad (4.1)$$

The principal \mathbb{C}^\times -bundle $\pi: \mathcal{Q} \rightarrow \mathcal{D}$ induces a principal \mathbb{C}^\times -orbi-bundle $[\pi]: [\mathcal{Q}/\Gamma] \rightarrow \mathbb{M}$ on $\mathbb{M} = [\mathcal{D}/\Gamma]$, since it is equivariant with respect to the natural action of $\Gamma = O(Q)^+$. The section (4.1) of π is not a section of $[\pi]$ since it is not Γ -equivariant. The line bundle associated with the principal \mathbb{G}_m -bundle $[\pi]: [\mathcal{Q}/\Gamma] \rightarrow \mathbb{M}$ will be denoted by $\mathcal{O}_{\mathbb{M}}(1)$.

The fixed locus \mathcal{Q}^g of an element $g \in \Gamma$ is a proper linear subspace of \mathcal{Q} . The element g is said to be a *reflection* if \mathcal{Q}^g is a hyperplane.

Lemma 4.1. *Any reflection in Γ is given by $z \mapsto z + (z, \delta) \cdot \delta$ for some $\delta \in \Delta(Q)$.*

Proof. Let $\delta \neq 0$ be a primitive element in $(Q^g)^\perp$. Then we have $g(\delta) = -\delta$ and g is given by $z \mapsto z - (2(z, \delta)/(\delta, \delta)) \cdot \delta$. Since Q is unimodular, there exists an element $z \in Q$ such that $(z, \delta) = 1$. Since $g(z) \in Q$, it follows that $2/(\delta, \delta) \in \mathbb{Z}$. Hence $(\delta, \delta) = \pm 2$. Since $g \in O^+(Q)$, we have $(\delta, \delta) = -2$. \square

Lemma 4.2. *The action of Γ on $\Delta(Q)$ is transitive.*

Proof. Let $\delta_1, \delta_2 \in \Delta(Q)$. By Nikulin's theory of discriminant forms of lattices, it follows that $\delta_i^\perp = \langle 2 \rangle \perp U \perp E_8$ and that there exists an element $g \in O(Q)$ such that $g(\delta_1) = \delta_2$. One can check that $O(\delta_i^\perp) \neq O^+(\delta_i^\perp)$. Hence we may assume that $g \in O^+(Q)$. \square

Let $H_Q = \cup_{\delta \in \Delta(Q)} \delta^\perp$ be the union of reflection hyperplanes $\delta^\perp = \{z \in Q \otimes \mathbb{C} \mid (z, \delta) = 0\}$, and $S_Q \subset Q$ be the locus where the stabilizer group is non-trivial and does not coincide with a group of order two generated by a reflection. The locus S_D contains not only intersections of more than two reflection hyperplanes, but also points where the corresponding K3 surface has a non-trivial automorphism. The definition of S_Q is chosen so that the action of Γ on $Q \setminus (H_Q \cup S_Q)$ is free, and the stabilizer of a point in $H_Q \setminus (H_Q \cap S_Q)$ is a group of order two generated by a reflection. Since the group Γ is countable, the locus H_Q is a countable union of hyperplanes, and S_Q is a countable union of linear subspaces of codimension at least two. The images of H_Q and S_Q in \mathcal{D} and M will be denoted by H_D , S_D , H_M , and S_M respectively.

Let Γ_2 be the index two subgroup of Γ generated by even numbers of reflections. This coincides with the kernel of the map $\det: \Gamma \rightarrow \{\pm 1\}$. Set $M_2 = \mathcal{D}/\Gamma_2$, $H_{M_2} = H_D/\Gamma_2$, and $S_{M_2} = S_D/\Gamma_2$. The map $M_2 \setminus S_{M_2} \rightarrow M \setminus S_M$ is a double cover branched along $H_M \setminus S_M$, and the map $\mathcal{D} \setminus S_D \rightarrow M_2 \setminus S_{M_2}$ is a universal cover since the action of Γ_2 on $\mathcal{D} \setminus S_D$ is free.

Let H_T and S_T be the inverse images of H_M and S_M by the period map $\Pi: T \xrightarrow{\sim} M$. Let further $\Delta_T \in \mathbb{C}[t]$ be the defining equation of H_T , and

$$V = \{([t], s) \in (T \setminus S_T) \times \mathbb{A}^1 \mid s^2 = \Delta_T(t)\} \quad (4.2)$$

be the double cover of $T \setminus S_T$ branched along $H_T \setminus S_T$. The restriction $\Pi|_{T \setminus S_T}: T \setminus S_T \xrightarrow{\sim} M \setminus S_M$ of the period map lifts to an isomorphism $\Pi_V: V \rightarrow M_2 \setminus S_{M_2}$, since both V and $M_2 \setminus S_{M_2}$ are the double covers of an isomorphic smooth varieties branched along smooth divisors which are identified under the isomorphism. This isomorphism is equivariant under the action of the covering transformation group $\Gamma/\Gamma_2 \cong \mathbb{Z}/2\mathbb{Z}$. The orbifold quotient $[V/(\mathbb{Z}/2\mathbb{Z})]$ is called the *root stack* of $T \setminus S_T$ of order 2 along $H_T \setminus S_T$.

The isomorphism Π_V can further be lifted to the isomorphism $\Pi_U: U \rightarrow \mathcal{D} \setminus S_D$, where U is the universal cover of V . This isomorphism is equivariant under the action of the covering transformation group $\text{Gal}((\mathcal{D} \setminus S_D) / (M \setminus S_M)) = \Gamma/\{\pm \text{id}\}$.

The pull-back of the principal \mathbb{C}^\times -bundle $\tilde{T} \rightarrow T$ by the composition $U \rightarrow V \rightarrow T \setminus S_T \hookrightarrow T$ will be denoted by $\tilde{U} \rightarrow U$. The action of the covering transformation group $\Gamma/\{\pm \text{id}\}$ lifts not to an action on \tilde{U} , but to an action of the central extension Γ of $\Gamma/\{\pm \text{id}\}$, in such a way that the lift $\Pi_{\tilde{U}}: \tilde{U} \rightarrow Q \setminus S_Q$ of the isomorphism $\Pi_U: U \rightarrow \mathcal{D} \setminus S_D$ is Γ -equivariant.

Proposition 4.3. *The isomorphism $\Pi_{\tilde{U}}: \tilde{U} \xrightarrow{\sim} Q \setminus S_Q$ is equivariant with respect to the natural \mathbb{C}^\times -action on both sides.*

Proof. Let $\varphi_{\tilde{U}}: \mathfrak{Y}_{\tilde{U}} \rightarrow \tilde{U}$ be the pull-back of $\varphi_{\tilde{T}}: \mathfrak{Y}_{\tilde{T}} \rightarrow \tilde{T}$. The element

$$\overline{\Omega} = \text{Res} \frac{\sum_{i=1}^4 (-1)^i q_i x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_4}{f}$$

of $H^0\left(\Omega_{\mathfrak{Y}/\tilde{T}}^2\right)$ gives a 2-form $\overline{\Omega}_t$ on \overline{Y}_t for each $t \in \tilde{T}$, whose pull-back to Y_t extends to a holomorphic 2-form Ω_t . Here we have set $(x_1, x_2, x_3, x_4) = (x, y, z, w)$ and $(q_1, q_2, q_3, q_4) = (6, 14, 21, 1)$.

Let $\mathfrak{L} \rightarrow \tilde{T} \setminus (H_{\tilde{T}} \cup S_{\tilde{T}})$ be the local system whose fiber over $t \in \tilde{T} \setminus (H_{\tilde{T}} \cup S_{\tilde{T}})$ is the second homology group $H_2(Y_t; \mathbb{Z})$. The monodromy of \mathfrak{L} along $H_{\tilde{T}}$ is the Picard-Lefschetz transformation with respect to the vanishing cycle C , which is the reflection along the homology class $[C]$ of the vanishing cycle. Note that the class $[C]$ is equal to the class of the (-2) -curve which

defines the reflection hyperplane. It follows that the pull-back of \mathfrak{L} to the double cover V does not have a monodromy along the ramification divisor H_V , and hence it extends to a local system \mathfrak{L}_V on V . The pull-back \mathfrak{L}_U of \mathfrak{L}_V to U is trivial since U is the universal cover of V . Let $\mathfrak{L}_{\tilde{U}}$ be the pull-back of \mathfrak{L}_U to \tilde{U} .

The period map $\Pi: T \rightarrow M$ is defined in such a way that the lift $\Pi_{\tilde{U}}: \tilde{U} \rightarrow \mathcal{D} \setminus S_{\mathcal{D}}$ is given by integration of Ω_t along a basis of $\mathfrak{L}_{\tilde{U}}$ obtained by choosing a global trivialization of $\mathfrak{L}_{\tilde{U}}$.

The action of $\alpha \in \mathbb{C}^\times$ on \tilde{T} sends a point $(t_i)_i$ to $(\alpha^i t_i)_i$. The change of f caused by this action can be absorbed by the coordinate change sending $w \mapsto \alpha^{-1}w$ and keeping x, y and z fixed. This sends Ω to $\alpha^{-1}\Omega$, so that the period will be multiplied by α^{-1} . This shows that $\Pi_{\tilde{U}}$ is \mathbb{C}^\times -equivariant, and Proposition 4.3 is proved. \square

One can rephrase the $\mathbb{C}^\times \times \Gamma$ -equivariant isomorphism in the language of orbifolds:

Corollary 4.4. *The restriction $\Pi|_{T \setminus S_T}: T \setminus S_T \xrightarrow{\sim} M \setminus M_T$ of the period map lifts to an isomorphism $[\Pi]: \mathbb{T} \setminus S_{\mathbb{T}} \xrightarrow{\sim} \mathbb{M} \setminus S_{\mathbb{M}}$ of orbifolds $\mathbb{T} \setminus S_{\mathbb{T}} = [\tilde{U}/(\mathbb{C}^\times \times \Gamma)]$ and $\mathbb{M} \setminus S_{\mathbb{M}} = [(\mathcal{Q} \setminus S_{\mathcal{Q}})/(\mathbb{C}^\times \times \Gamma)]$ in such a way that $[\Pi]^* \mathcal{O}_{\mathbb{M} \setminus S_{\mathbb{M}}}(1) \cong \mathcal{O}_{\mathbb{T} \setminus S_{\mathbb{T}}}(1)$.*

5 The canonical bundle on the moduli space

Line bundles on $\mathbb{M} = [\mathcal{Q}/\mathbb{C}^\times \times \Gamma]$ are $\mathbb{C}^\times \times \Gamma$ -equivariant line bundles on \mathcal{Q} . Since \mathcal{Q} is Stein, the Picard group $\text{Pic } \mathbb{M}$ is isomorphic to the group $\text{Char}(\mathbb{C}^\times \times \Gamma)$ of characters. We write a line bundle corresponding to the character $\mathbb{C}^\times \times \Gamma \ni (\alpha, g) \mapsto \alpha^{-k} \cdot (\det g)^l$ as $\mathcal{O}_{\mathbb{M}}(k) \otimes \det^{k+l}$. Although T^* is a Fano variety, the orbifold \mathbb{T} has an ample canonical bundle:

Proposition 5.1. *The canonical bundle on \mathbb{M} is given by $\omega_{\mathbb{M}} \cong \mathcal{O}_{\mathbb{M}}(10) \otimes \det$.*

Proof. Recall that \mathcal{D} is an open subset defined by $(\Omega, \overline{\Omega}) > 0$ of a quadratic hypersurface in $\mathbb{P}(Q \otimes \mathbb{C})$ defined by $(\Omega, \Omega) = 0$. The canonical bundle of a degree k hypersurface X in \mathbf{P}^n is given by $\mathcal{O}_X(n+1-k)$. Since the definition of the weight of the \mathbb{C}^\times -action on a line bundle on $[\mathcal{Q}/\mathbb{C}^\times]$ is inverse to the usual convention, one has $\omega_{\mathcal{D}} \cong \mathcal{O}_{\mathcal{D}}(10)$. Since a reflection changes the sign of a top differential form, one concludes that $\omega_{\mathbb{M}} \cong \mathcal{O}_{\mathbb{M}}(10) \otimes \det$. \square

The following proposition concludes the proof of Theorem 1.2:

Proposition 5.2. *One has $\deg \Delta_T = 504$.*

Proof. Let d be the degree of Δ_T , which must be even since all weights in \mathbf{w} is even. Since we deal only with line bundles, we ignore subsets of codimension greater than one to simply notation. Let $p: V \rightarrow T$ be the double cover branched along H_T . Then one has $\omega_V \cong p^* \omega_T \otimes \mathcal{O}_V(H_V)$, where H_V is the ramification divisor. One has $\omega_T \cong \mathcal{O}_T(-|\mathbf{w}|)$ where $|\mathbf{w}| = w_1 + \dots + w_{11} = 242$.

Let $\mathbb{V} = [V/G]$ be the orbifold quotient of the double cover V of T by the covering transformation group $G = \text{Gal}(V/T) \cong \mathbb{Z}/2\mathbb{Z}$. The morphism $p: V \rightarrow T$ induces a morphism $[p]: \mathbb{V} \rightarrow T$. It follows from the definition that the Picard group $\text{Pic } \mathbb{V}$ of \mathbb{V} is given by the G -equivariant Picard group $\text{Pic}^G T$ of T . It is generated by two elements $[p]^* \mathcal{O}_T(1)$ and $\mathcal{O}_{\mathbb{V}}(H_{\mathbb{V}})$ with one relation $[p]^* \mathcal{O}_T(d) \cong \mathcal{O}_{\mathbb{V}}(2H_{\mathbb{V}})$. One has an isomorphism $\text{Pic } \mathbb{V} \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$, where \mathbb{Z} is generated by $[p]^* \mathcal{O}_T(1)$ and $\mathbb{Z}/2\mathbb{Z} \cong \text{Char}(G)$ is generated by $[p]^* \mathcal{O}_T(-d/2) \otimes \mathcal{O}_{\mathbb{V}}(H_{\mathbb{V}})$.

We adopt a notation parallel to that of $\text{Pic } \mathbb{M}$ and write $\mathcal{O}_{\mathbb{V}}(k) \otimes \det^l = [p]^* \mathcal{O}_T(k) \otimes \mathcal{O}_{\mathbb{V}}(1) \otimes ([p]^* \mathcal{O}_T(-d/2) \otimes \mathcal{O}_{\mathbb{V}}(H_{\mathbb{V}}))^{k+l}$. Then one has

$$\begin{aligned} \omega_{\mathbb{V}} &= [p]^* \omega_T \otimes \mathcal{O}_{\mathbb{V}}(H_{\mathbb{V}}) \\ &= [p]^* \mathcal{O}_T(-242) \otimes \mathcal{O}_{\mathbb{V}}(H_{\mathbb{V}}) \\ &= [p]^* \mathcal{O}_T(-242 + d/2) \otimes ([p]^* \mathcal{O}_T(-d/2) \otimes \mathcal{O}_{\mathbb{V}}(H_{\mathbb{V}})) \\ &= \mathcal{O}_{\mathbb{V}}(10) \otimes \det, \end{aligned}$$

so that $-242 + d/2 = 10$ and $d = 504$. □

6 Discriminant and resultant

We prove (6.2) in this section. The discriminant of $y^3 + g_2(x, w; t)y + g_3(x, w; t)$ as a polynomial of y is given by $4g_2(x, w; t)^3 + 27g_3(x, w)^2$, which defines a hypersurface of degree 84 in $\mathbb{P}(6, 1) = \mathbb{P}\text{roj } \mathbb{C}[x, w]$. In other words, $[4g_2(x, w; t)^3 + 27g_3(x, w)^2]/w^{84}$ is a polynomial of degree 14 in x/w^6 . Let $k(t)$ be the discriminant of this polynomial in one variable, which is a homogeneous polynomial of degree $14 \cdot 13 \cdot 6 = 1092$. A general point on the divisor $D \subset \mathbf{P}(\mathbf{w})$ defined by k corresponds to the locus where two fibers of Kodaira type I_1 collapses into one fiber. The divisor D is a linear combination of two prime divisors D_1 and D_2 . A general point on the component D_1 corresponds to the case when neither g_2 nor g_3 vanishes at p , and a general point on the other component D_2 corresponds to the case when both g_2 and g_3 vanishes at p . In the former case, the resulting singular fiber is of Kodaira type I_2 , and the surface \overline{Y}_t acquires an A_1 -singularity. In the latter case, the resulting singular fiber is of Kodaira type II , and the surface \overline{Y}_t does not acquire any new singularity. The defining equation of D_1 is Δ_T . The defining equation of D_2 is the resultant of g_2 and g_3 , which is given as the determinant

$$f_2 = \begin{vmatrix} t_{28} & t_{22} & t_{16} & t_{10} & t_4 & & & & & & \\ & t_{28} & t_{22} & t_{16} & t_{10} & t_4 & & & & & \\ & & t_{28} & t_{22} & t_{16} & t_{10} & t_4 & & & & \\ & & & t_{28} & t_{22} & t_{16} & t_{10} & t_4 & & & \\ & & & & t_{28} & t_{22} & t_{16} & t_{10} & t_4 & & \\ & & & & & t_{28} & t_{22} & t_{16} & t_{10} & t_4 & \\ & & & & & & t_{28} & t_{22} & t_{16} & t_{10} & t_4 \\ t_{40} & t_{36} & t_{30} & t_{24} & t_{18} & t_{12} & 0 & 1 & & & \\ & t_{40} & t_{36} & t_{30} & t_{24} & t_{18} & t_{12} & 0 & 1 & & \\ & & t_{40} & t_{36} & t_{30} & t_{24} & t_{18} & t_{12} & 0 & 1 & \\ & & & t_{40} & t_{36} & t_{30} & t_{24} & t_{18} & t_{12} & 0 & 1 \end{vmatrix} \quad (6.1)$$

of the Sylvester matrix, which is homogeneous of degree $d_2 = 196$.

Lemma 6.1. *For a polynomial $f(x, y, t) = y^3 + g_2(x, t)y + g_3(x, t)$ in three variables, let $h(x, t) = 4g_2(x, t)^3 + 27g_3(x, t)^2$ be the discriminant of $f(x, y, t)$ as a polynomial of y , $k(t)$ be the discriminant of $h(x, t)$ as a polynomial of x , and $r(t)$ be the resultant of the pair $(g_2(x, t), g_3(x, t))$ as polynomials of x . Then one has $k(t) = r(t)^3 \cdot \ell(t)$ for some polynomial $\ell(t) \in \mathbb{C}[t]$.*

Proof. Introduce variables $(\alpha_i)_{i=1}^n$ and $(\beta_i)_{i=1}^m$ and set $g_2 = \prod_{i=1}^n (x - \alpha_i)$ and $g_3 = \prod_{i=1}^m (x - \beta_i)$. Then one has $r = \prod_{i=1}^n \prod_{j=1}^m (\alpha_i - \beta_j)$, which has a zero of order one along $\alpha_i = \beta_j$ for each

pair (i, j) . Hence it suffices to show that the order of zero of the discriminant k along $\alpha_i = \beta_j$ is 3. For this purpose, we may assume $n = m = 1$ and consider the case when $g_2 = x - \alpha$ and $g_3 = x - \beta$, since other variables do not contribute at the generic point of the divisor $\alpha_i = \beta_j$. Then one has $h = 4(x - \alpha)^3 + 27(x - \beta)^2$. By replacing $x - \alpha$ with x , one obtains $h = 4x^3 + 27(x + \alpha - \beta)^2$, whose discriminant is given by $k = (\alpha - \beta)^4 - (\alpha - \beta)^3$. The order of zero of k along $\alpha = \beta$ is precisely 3, and Lemma 6.1 is proved. \square

Since $\deg D - 3 \deg D_2 = 1092 - 3 \cdot 196 = 504 = \deg D_1$, one has $D = D_1 + 3D_2$ and

$$\Delta_T(t) = \frac{k(t)}{r(t)^3}. \quad (6.2)$$

Remark 6.2. In general, if we set $h(x, t) = g_2(x, t)^n - g_3(x, t)^m$ for $n \geq m$, then the order of vanishing of the discriminant $k(t)$ of $h(x, t)$ along the resultant $r(t)$ of $g_2(x, t)$ and $g_3(x, t)$ is given by $n(m - 1)$. This follows from the fact that the set of solutions of

$$(x - \alpha)^n - (x - \beta)^m = 0 \quad (6.3)$$

for $n \geq m$ near $\alpha = \beta$ consists of m solutions of the form

$$a_i = \beta + \zeta_m^i (\beta - \alpha)^{n/m} + o((\beta - \alpha)^{n/m}), \quad i = 0, \dots, m - 1 \quad (6.4)$$

and $n - m$ solutions of the form

$$b_j = \alpha + \zeta_{n-m}^j (\beta - \alpha) + o(\beta - \alpha), \quad j = 0, \dots, n - m - 1, \quad (6.5)$$

so that the leading term of the discriminant

$$\left(\prod_{i < i'} (a_i - a_{i'})^2 \right) \cdot \left(\prod_{j < j'} (b_j - b_{j'})^2 \right) \cdot \left(\prod_{i, j} (a_i - b_j)^2 \right) \quad (6.6)$$

is given by

$$\prod_{i < i'} (a_i - a_{i'})^2 \sim ((\beta - \alpha)^{n/m})^{m(m-1)} = (\beta - \alpha)^{n(m-1)}. \quad (6.7)$$

7 Holomorphic extension of the period map

In this section, we give a proof of Proposition 7.1, which completes a proof of Theorem 1.1. The original proof in [Shi] relies on the study of asymptotic behavior of the period map.

Proposition 7.1 ([Shi, Proposition 5.2]). *The period map $\Pi: T \rightarrow M$ extends to a biregular isomorphism $\Pi^*: T^* \rightarrow M^*$ with the Satake-Baily-Borel compactification M^* of M .*

Proof. The inclusions $\iota_T: T \hookrightarrow T^*$ and $\iota_M: M \hookrightarrow M^*$ induces isomorphisms $\iota_T^*: \text{Pic } T^* \xrightarrow{\sim} \text{Pic } T$ and $\iota_M^*: \text{Pic } M^* \xrightarrow{\sim} \text{Pic } M$ of Picard groups since both $T^* \setminus T$ and $M^* \setminus M$ have codimensions greater than one. One has $\text{Pic } T = \mathbb{Z}$, and the total coordinate ring $\bigoplus_{\mathcal{L} \in \text{Pic } T} H^0(\mathcal{L})$ is isomorphic to the weighted polynomial ring of weight (3.5). This is isomorphic to the total coordinate ring of M since T is isomorphic to M . Since the space $H^0(\mathcal{O}_M(k))$ of sections of $\mathcal{O}_M(k)$ is the space $A_k(\Gamma, \mathbf{1})$ of modular forms of weight k with the trivial character $\mathbf{1} \in \text{Char}(\Gamma)$, the total coordinate ring of M is given by the ring $A(\Gamma, \mathbf{1}) = \bigoplus_{k=0}^{\infty} A_k(\Gamma, \mathbf{1})$ of modular forms with the trivial character. Since M^* is isomorphic to $\text{Proj } A(\Gamma, \mathbf{1})$ by [BB66, Theorem 10.11], one obtains $M^* = T^*$. \square

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